Particle in a ring:-

① The schrodinger equation and wave function:-

Consider a particle of mass m rotating in a circle of radius r in xy plane. If the potential energy (V) of the particle is zero (i.e. its unacted upon by any external force), the Hamiltonian in the simple form can be written as,

$$\hat{H} = \frac{-h^2}{8\pi^2 m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \quad \text{— ①}$$

since the motion is circular, it will be more convinient to express $\hat{H}$ in polar coordinates (r, ϕ). The Cartesian coordinates x ~~and~~ and y are related to polar coordinates r and ϕ as,

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \quad \text{— ②}$$

where $\phi \rightarrow$ angular velocity variable which varies from 0 to 2π, radius, r, is constant. Hence, eq ① in polar coordinates become,

$$\hat{H} = \frac{-h^2}{8\pi^2 m r^2}\left(\frac{\partial^2}{\partial \phi^2}\right) = \frac{-h^2}{8\pi^2 I}\left(\frac{\partial^2}{\partial \phi^2}\right) \quad \text{— ③}$$

where $I = mr^2$ is the moment of inertia.

The eigenfunctions $F(\phi)$, or 'F', will be a function of ϕ only.

Hence, the schrodinger wave equation becomes.

$$\frac{-h^2}{8\pi^2 I} \cdot \frac{d^2F}{d\phi^2} = EF \quad \text{—— (4)}$$

or $$\frac{d^2F}{d\phi^2} + M^2F = 0 \quad \text{—— (5)}$$

where $$M^2 = \frac{8\pi^2 IE}{h^2} \quad \text{—— (6)}$$

The equation (5) can be solved as

$$\left.\begin{aligned} F &= N \sin M\phi \\ F' &= N' \cos M\phi \end{aligned}\right\} \text{— (7)}$$

as the real set, and

$$F'' = A \exp(\pm iM\phi) \text{ — (8)}$$

as the imaginary set.

The trignometric forms are also called <u>circular harmonics</u>.

Using eq. (7) and (8) are equivalent in the view of theorem.

$$\exp(\pm iM\phi) = \cos M\phi \pm i\sin M\phi \text{ — (9)}$$

The functions (7) and (8) are finite and continuous for all values of ϕ and M. They are single valued since angles ϕ and $\phi + 2\pi$ represents the same point, and the property of single valuedness demands that

$$F(\phi) = F(\phi + 2\pi).$$

Using the real set, we get,

$$\sin M\phi = \sin M(\phi + 2\pi)$$
$$\cos M\phi = \cos M(\phi + 2\pi)$$

This is possible only if $M = 0, \pm 1, \pm 2 \ldots\ldots$

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$$A \exp(iM\phi) = A \exp(iM(\phi + 2\pi)) = A \exp(iM\phi).\exp(i2\pi M)$$

$$\exp(i2\pi M) = 1$$

or, $\cos 2\pi M + i \sin 2\pi M = 1$

only if $M = 0, \pm 1, \pm 2$

As regards boundary conditions, there is no barrier to the particles motion as long it ~~has~~ is on the ring, and so there is no condition for the wave function to vanish at any point. ~~(For M = 0,~~

(For $M=0$, $N \sin M\phi = 0$ but $N' \cos M\phi = N'$)

* show that $F = N \sin\frac{\phi}{2}$ is not acceptable even though

$$\sin\frac{\phi}{2} = \sin\frac{1}{2}(\phi + 2\pi)$$

let $F = \sin\frac{\phi}{2}$ and $F' = \sin\frac{1}{2}(\phi + 2\pi)$

then $\frac{dF}{d\phi} = \frac{1}{2}\cos\frac{\phi}{2}$ and

$$\frac{dF'}{d\phi} = \frac{1}{2}\cos\left(\frac{\phi}{2} + \pi\right)$$

$$= \frac{1}{2}\left(\cos\frac{\phi}{2}\cos\pi - \sin\frac{\phi}{2}\sin\pi\right)$$

$$= -\frac{1}{2}\cos\frac{\phi}{2}$$

since $\frac{dF}{d\phi} \neq \frac{dF'}{d\phi'}$, there is discontinuity.

Normalization and Orthogonality :-

The normalization factor, N, can be determined as

$$\int_0^{2\pi} (N \sin M\phi)^2 \, d\phi = 1$$

or $$N^2 \int_0^{2\pi} \sin^2 M\phi \, d\phi = N^2 \int_0^{2\pi} \left(\frac{1 - \cos 2M\phi}{2} \right) d\phi = 1$$

or $$N^2 . \pi = 1 \quad \text{or} \quad N = \frac{1}{\sqrt{\pi}}$$

similarly $$N' = \frac{1}{\sqrt{\pi}}$$

The normalized real set of wave functions are

$$F = \frac{1}{\sqrt{\pi}} \sin M\phi \quad \text{and} \quad F = \frac{1}{\sqrt{\pi}} . \cos M\phi$$

For $M = 0$, $F' = \frac{1}{\sqrt{\pi}}$

For the imaginary set,

$$\int_0^{2\pi} A^* \exp \pm (iM\phi) , A \exp(\pm iM\phi) \cdot d\phi = 1$$

or $$\int_0^{2\pi} |A|^2 d\phi = 1 \quad \text{or} \quad |A|^2 . 2\pi = 1 \quad \text{or} \quad |A| = \frac{1}{\sqrt{2\pi}}$$

Hence $$F'' = \frac{1}{\sqrt{2\pi}} \exp(\pm iM\phi)$$

For $M = 0$, $F'' = \frac{1}{\sqrt{2\pi}}$

It is not difficulty to show that the functions are orthogonal sets as for example

$$\frac{1}{\pi} \int_0^{2\pi} \sin\phi \cdot \cos\phi \, d\phi = 0.$$